Finite Math - Fall 2016

Lecture Notes - 10/3/2016

Section 4.1 - Systems of Linear Equations in Two Variables

Solving Using Elimination. We now turn to a method that, unlike graphing and substitution, is generalizable to systems with more than two variables easily. There are a set of rules to follow when doing this

Theorem 1. *A system of linear equations is transformed into an equivalent system if*

(1) *two equations are interchanged*

(2) *an equation is multiplied by a nonzero constant*

(3) *a constant multiple of one equation is added to another equation.*

Example 1. *Solve the following system using elimination*

$$\begin{array}{rcrcr} 3x & - & 2y & = & 8 \\ 2x & + & 5y & = & -1 \end{array}$$

Solution. *If we subtract the second equation from the first one, we end up with the new system*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ 2x & + & 5y & = & -1 \end{array}$$

Now, we can subtract 2 times the first equation ($2x - 14y = 18$) from the second equation to get

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & 19y & = & -19 \end{array}$$

Now we divide the second equation by 19 to get

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & y & = & -1 \end{array}$$

and finally, we will add 7 times the second equation ($7y = -7$) to the first equation

$$\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & -1 \end{array}$$

This gives the answer of $x = 2, y = -1$.

We could have also used a combination of substitution and elimination above, for example, once we knew that $y = -1$, we could have just plugged that into the first equation, but this solution was a little preview for the later sections.

Example 2. *Solve the system using elimination*

$$\begin{array}{rcrcl} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Solution. $x = 2, y = -1$

We now want to look at the case when the system does not have one unique solution, but is either inconsistent or is consistent but dependent.

Example 3. *Solve the system*

$$\begin{array}{rcrcl} 2x & + & 6y & = & -3 \\ x & + & 3y & = & 2 \end{array}$$

Solution. *We begin by subtracting twice the second equation from the first*

$$\begin{array}{rcrcl} & & 0 & = & -7 \\ x & + & 3y & = & 2 \end{array}$$

The first equation has become $0 = -7$ which is obviously untrue. This is an example of what happens when the system is inconsistent*.*

Example 4. *Solve the system*

$$\begin{array}{rcrcl} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Solution. *First, let's multiply the first equation by 2 to get rid of the fraction*

$$\begin{array}{rcrcl} 2x & - & y & = & 8 \\ -2x & + & y & = & -8 \end{array}$$

Now, if we add the two equations together, we get

$$0 = 0$$

which is always true. This means that the two equations are the same equation, just one is (maybe) multiplied by a constant. This is a consistent but dependent *system of equations. If we let $x = k$, where k is any real number, then we get that $y = 2k - 8$. So, for any k, $(k, 2k - 8)$ is a solution. In this case, the variable k is called a* parameter*.*

Example 5. *Solve the systems*

(a)

$$\begin{array}{rcrcl} 5x & + & 4y & = & 4 \\ 10x & + & 8y & = & 4 \end{array}$$

(b)

$$\begin{array}{rcrcl} 6x & - & 5y & = & 10 \\ -12x & + & 10y & = & -20 \end{array}$$

Applications. There are a variety of applications of systems of equations. For a simple example, consider the following

Example 6. *Dennis wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains* 3 *grams of protein and* 15 *milligrams of calcium. An ounce of yogurt contains* 1 *gram of protein and* 41 *milligrams of calcium. How many ounces of cottage cheese and yogurt should Dennis eat each day to provide exactly* 62 *grams of protein and* 760 *milligrams of calcium?*

Solution. *We begin by setting up an equation for the amount of protein consumed and another equation for the amount of calcium consumed. Suppose Dennis eats x ounces of cottage cheese and y ounces of yogurt. Since* 1 *ounce of cottage cheese contains* 3 *grams of protein we know that x ounces of cottage cheese will contain $3x$ grams of protein; likewise y ounces of yogurt contains y grams of protein. Since Dennis wants to consume a total of* 62 *grams of protein, we get the equation*

$$3x + y = 62$$

We similarly set up an equation for milligrams of calcium consumed: x ounces of cottage cheese has $15x$ milligrams of calcium and y ounces of yogurt has 41 *milligrams of calcium, and Dennis wants to eat exactly* 760 *milligrams of calcium, so*

$$15x + 41y = 760$$

Thus we have a system of equations to solve which will tell us exactly how much cottage cheese and yogurt Dennis should eat

$$\begin{array}{rcrcr} 3x & + & y & = & 62 \\ 15x & + & 41y & = & 760 \end{array}$$

If we multiply the top equation by -5 then add the two equations together, we get

$$36y = 450$$

giving

$$y = 12.5$$

Plugging this into the first equation gives us

$$3x + 12.5 = 62 \iff 3x = 49.5 \iff x = 16.5.$$

So, Dennis should eat 16.5 *grams of cottage cheese and* 12.5 *grams of yogurt each day to reach his target.*

Example 7. *A fruit grower uses two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains* 8 *pounds of nitrogen and* 4 *pounds of phosphoric acid. Each bag of brand B contains* 7 *pounds of nitrogen and* 6 *pounds of phosphoric acid. Tests indicate that the grove needs* 720 *pounds of nitrogen and* 500 *pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid?*

Solution. 41 *bags of brand A and* 56 *bags of brand B.*

In a free market economy, the price of a product is determined by the relationship between supply and demand. Suppliers are more willing to supply product when the price is high and consumers have a higher demand for a product when the price is low. In a free competitive market, the price tends to move toward an *equilibrium price*, where the supply and demand are equal. The supply and demand at that price is called the *equilibrium quantity*. Graphically, this will be the intersection of the supply and demand curves (which we will assume to be lines here).

Example 8. *At a price of* $\$1.88$ *per pound, the supply for cherries in a large city is* $16,000$ *pounds and the demand is* $10,600$ *pounds. When the price drops to* $\$1.46$ *per pound, the supply decreases to* $10,000$ *pounds and the demand increases to* $12,700$ *pounds. Assume that the price-supply and price-demand equations are linear.*

(a) *Find the price-supply equation.*

(b) *Find the price-demand equation.*

(c) *Find the supply and demand at a price of* $\$2.09$ *per pound.*

(d) *Find the supply and demand at a price of* $\$1.32$ *per pound.*

(e) *Find the equilibrium price and equilibrium demand.*

Solution. *Let* x *be the quantity in thousands of pounds and let* p *be the price. We will write points as* (x, p).

(a) *When the price is* $\$1.88$ *the supply is* $16,000$, *so we have the point* $(16, 1.88)$; *and when the price is* $\$1.46$ *the supply is* $10,000$, *giving the point* $(10, 1.46)$. *Using the point-slope formula for a line, we can find the equation for the supply curve:*

$$p - 1.88 = \frac{1.46 - 1.88}{10 - 16}(x - 16) = 0.07(x - 16)$$

or,

$$p = 0.07x + 0.76.$$

(b) *When the price is* $\$1.88$ *the demand is* $10,600$, *so we have the point* $(10.6, 1.88)$; *and when the price is* $\$1.46$ *the demand is* $12,700$, *giving the point* $(12.7, 1.46)$. *Using the point-slope formula for a line, we can find the equation for the demand curve:*

$$p - 1.88 = \frac{1.46 - 1.88}{12.7 - 10.6}(x - 10.6) = -0.2(x - 10.6)$$

or,

$$p = -0.2x + 4.$$

(c) *Here, we just plug in $p = 2.09$ to our equations and find x:*
The supply will be

$$2.09 = 0.07x + 0.76 \quad \Longrightarrow \quad 0.07x = 1.33$$

so the supply is

$$x = 19$$

or $19,000$ *pounds.*
The demand will be

$$2.09 = -0.2x + 4 \quad \Longrightarrow \quad 0.2x = 1.91$$

so the demand is

$$x = 9.55$$

or $9,550$ *pounds.*

(d) *We do the same thing as in the previous part, just plugging in $p = 1.32$ instead.*
The supply will be

$$1.32 = 0.07x + 0.76 \quad \Longrightarrow \quad 0.07x = 0.56$$

so the supply is

$$x = 8$$

or $8,000$ *pounds. Notice that the supply went down as the price decreased.*
The demand will be

$$1.32 = -0.2x + 4 \quad \Longrightarrow \quad 0.2x = 2.68$$

so the demand is

$$x = 13.4$$

or $13,400$ *pounds. Notice that the demand went up with the decreased price.*

(e) *To find the equilibrium, we need to solve the system of equations determined by the supply and demand equations. Since both are already solved for p, we will use the substitution method to get*

$$0.07x + 0.76 = -0.2x + 4 \quad \Longrightarrow \quad 0.27x = 3.24$$

giving

$$x = 12$$

Thus the equilibrium quantity is $12,000$ *pounds, and the equilibrium price is (plug x into either equation)*

$$p = -0.2(12) + 4 = 1.60$$

that is, $\$1.60$ *per pound.*